

Yang-Mills Theory as an Illustration of the Covariant Quantization of Superstrings

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We present a new approach to the quantization of the superstring. After a brief review of the classical Green-Schwarz formulation for the superstring and Berkovits' approach to its quantization based on pure spinors, we discuss our formulation without pure spinor constraints. In order to illustrate the ideas on which our work is based, we apply them to pure Yang-Mills theory. In the appendices, we include some background material for the Green-Schwarz and Berkovits formulations, in order that this presentation be self contained.

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1. Introduction

String theory is mostly based on the Ramond-Neveu-Schwarz (RNS) formulation, with worldsheet fermions ψ^m in the vector representation of the spacetime Lorentz group $SO(9,1)$. This formulation exhibits classically a $N = 1$ local supersymmetry of the worldsheet. The BRST symmetry of the RNS formulation is based on the super-reparametrization invariance of the worldsheet. The fundamental fields are the bosons x^m , the fermions ψ^m , the reparametrization ghosts b_{zz}, c^z and the superghosts β, γ . Physical states correspond to vertex operators which i) belong to the BRST cohomology and ii) are annihilated by b_0 for the open string, or by b_0 and \tilde{b}_0 for the closed string. To obtain a set of physical states which form a representation of spacetime supersymmetry, the GSO projection is applied to remove half of the physical states. Spacetime supersymmetry is thus not manifest, and the study of Ramond-Ramond backgrounds is not feasible. Therefore, one would prefer a formulation with spacetime fermions θ^α belonging to a representation of $Spin(9,1)$ because it would keep spacetime supersymmetry (susy) manifest. At the classical level, such a formulation was constructed by Green and Schwarz in 1984 [1]. Their classical action contains two fermions $\theta^{i\alpha}$'s ($i = 1, 2$) and the bosonic coordinates x^m . Each of the θ 's is real and can be chiral or anti-chiral (type IIA/B superstrings): they are 16-component Majorana-Weyl spinors which are spacetime spinors and worldsheet scalars. We shall denote chiral spinors by contravariant indices θ^α with $\alpha = 1, \dots, 16$; antichiral spinors are denoted by θ_α , also with $\alpha = 1, \dots, 16$.

The rigid spacetime supersymmetry is given by the usual non-linear coordinate representation

$$\delta_\epsilon \theta^{i\alpha} = \epsilon^{i\alpha}, \quad \delta_\epsilon x^m = \bar{\epsilon}^i \Gamma^m \theta^i = i \epsilon^{i\alpha} \gamma_{\alpha\beta}^m \theta^{i\beta}, \quad (1.1)$$

where $\gamma_{\alpha\beta}^m$ are real symmetric 16×16 matrices and the flavor indices $i = 1, 2$ are summed over. (In appendix A, Dirac matrices and Majorana-Weyl spinors are reviewed). Susy-invariant building blocks are

$$\Pi_\mu^m \equiv \partial_\mu x^m - i \sum_{i=1}^2 \theta^i \gamma^m \partial_\mu \theta^i, \quad \partial_\mu \theta^{i\alpha} \quad (1.2)$$

where $\mu = 0, 1$ and $\partial_0 = \partial_t$ and $\partial_1 = \partial_\sigma$. A natural choice for the action on a flat background spacetime and curved worldsheet would seem to be

$$\mathcal{L} = \frac{1}{2\pi} \sqrt{-h} h^{\mu\nu} \eta_{mn} \Pi_\mu^m \Pi_\nu^n, \quad (1.3)$$

with $h^{\mu\nu}$ the worldsheet metric, because it is the susy-invariant line element (a generalization of the action for the bosonic string). However, it yields no kinetic term for the fermions. Even if one could produce a kinetic term, there would still be the problem that one would have $\frac{1}{2}(16 + 16) = 16$ fermionic propagating modes and 8 bosonic propagating modes. Such a theory could not yield a linear representation of supersymmetry.

A resolution of this problem became possible when Siegel found a new local fermionic symmetry (κ -symmetry) for the point particle [2]. Green and Schwarz tried to find this symmetry in their string, and they discovered that it is present, but only after adding a Wess-Zumino-Novikov-Witten term to the action. Using this symmetry one could impose the gauge $\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0$ (where $\Gamma^\pm = \Gamma^0 \pm \Gamma^9$), and if one then also fixed the local scale and general coordinate symmetry by $h^{\mu\nu} = \eta^{\mu\nu}$, and the remaining conformal symmetry by $x^+(\sigma, t) = x_0^+ + p^+t$, the action became a free string theory with 8 fermionic degrees of freedom and 8 bosonic degrees of freedom. Susy was linearly realized and quantization posed no problem.

However, in this combined κ -light cone gauge, manifest $SO(9, 1)$ Lorentz invariance is lost, and with it all the reasons for studying the superstring in the first place. (We shall call the string of Green and Schwarz the superstring, to distinguish it from the RNS string which we call the spinning string.)

Going back to the original classical action, it was soon realized that second class constraints were present, due to the definition of the conjugate momenta of the θ 's. These second class constraints could be handled by decomposing them w.r.t. a non-compact $SU(5)$ subgroup of $SO(9, 1)$ (see appendix D), but then again manifest Lorentz invariance was lost. An approach to quantization which could deal with second class constraints and keep covariance was needed. By using a proposal of Faddeev and Fradkin to add further fields, one could turn second class constraints into first class constraints, but upon quantization one now obtained an infinite set of ghosts-for-ghosts, and problems with the calculation of anomalies were encountered. At the end of the 80's, several authors tried different approaches, but they always encountered infinite sets of ghosts-for-ghosts, and 15 years of pain followed [3].

A few years ago Berkovits developed a new line of thought [4]. Taking a flat background and a flat worldsheet metric, the central charge c of 10 free bosons x^m and one θ is $c = 10 - 2 \times 16 = -22$ (there is a conjugate momentum $p_{z\alpha}$ for θ^α). He noted that if one decomposes a chiral spinor λ^α under the non-compact $SU(5)$ subgroup of $SO(9, 1)$, it

decomposes as $\underline{16} \rightarrow \underline{10} + \underline{5}^* + \underline{1}$ (see Appendix D). Imposing the constraint

$$\lambda^T \gamma^m \lambda = 0, \quad (1.4)$$

also known as *pure spinor constraint*, one can express the $\underline{5}^*$ in terms of the $\underline{10}$ and $\underline{1}$, and hence it seemed that by adding a commuting pure spinor (with conjugate momenta for the $\underline{10}$ and $\underline{1}$), one could obtain vanishing central charge: $c = 10_x - 2 \times 16_{\theta, p_\theta} + 2 \times (10+1)_{\lambda, p_\lambda} = 0$. In the past few years, he has developed this approach further.

Having a constraint such as (1.4) in a theory leads to problems at the quantum level in the computation of loop corrections and in the definition of path integral. A similar situation occurred in superspace formulation of supergravity, where one must impose constraints on the supertorsions; in that case the constraints were solved and the covariance was sacrificed. One could work only with $\underline{10}$ and $\underline{1}$, but then one would again violate manifest Lorentz invariance.

We have developed an approach [5] which starts with the same $\theta^\alpha, p_{z\alpha}$ and λ^α as used by Berkovits, but we relax the constraint (1.4) by adding new ghosts. In Berkovits' and our approach one has the BRST law $s\theta^\alpha = i\lambda^\alpha$, with real θ^α , but in Berkovits' approach λ^α must be complex in order that (1.4) have a solution at all, *whereas in our approach λ^α is real*. The law $s\theta^\alpha = i\lambda^\alpha$ is an enormous simplification over the law one would obtain from the κ -symmetry law $\delta_\kappa \theta^\alpha = \Pi_\mu^m (\gamma_m \kappa^\mu)^\alpha$ with selfdual $\kappa^{\mu\alpha}$. It is this simpler starting point that avoids the infinite set of ghosts-for-ghosts. First, we give a brief review of the classical superstring action from which we shall only extract a set of first class constraints $d_{z\alpha}$. These first class constraints are removed from the action and used to construct a BRST charge.

We deduce the full theory by requiring nilpotency of the BRST charge: each time nilpotency on a given field does not hold we add a new field (ghost) and define its BRST transformation rule such that nilpotency holds. A priori, one might expect that one would end up again with an infinite set of ghosts-for-ghosts, but to our happy surprise the iteration procedure stops after a finite number of steps.

In some modern approaches the difference between the action and the BRST charge becomes less clear (in the BV formalism the action is even equal to the BRST charge). So the transplantation of the first class constraints from the action to the BRST charge may not be as drastic as it may sound at first. We may in this way create a different off-shell formulation of the same physical theory. The great advantage of this procedure is that one

is left with a free action, so that propagators become very easy to write down, and OPE's among vertex operators become as easy as in the RSN approach.

We shall now present our approach. We have a new definition of physical states, and we obtain the correct spectrum for the open string as well as for the closed superstring, both at the massless level and at the massive levels. Since these notes are intended as introduction to our work, we give much background material in the appendices. Such material is not present in our papers, but may help to understand the reasons and the technical aspects of our approach.

We have found since the conference some deep geometrical meanings of the new ghosts, but we have not yet found the underlying classical action to which our quantum theory corresponds. Sorokin, Tonin and collaborators have recently shown [6] how one can obtain Berkovits' theory from a $N = (2, 0)$ worldsheet action with superdiffeomorphism embeddings, and it is possible that a similar approach yields our theory.

2. The classical Green-Schwarz action

As we already mentioned, a natural generalization of the bosonic string with $\mathcal{L} \sim (\partial_\alpha x^m)^2$ with spacetime supersymmetry is the supersymmetric line element given in (1.2) and (1.3). If one considers the interaction term $\partial_\mu x^m (\theta \gamma_m \partial^\mu \theta)$ and if one chooses the light cone gauge $x^+ = x_0^+ + p^+ t$ one obtains a term $p^+ \theta \gamma_+ \partial_t \theta = (\sqrt{p^+} \theta) \gamma_+ \partial_t (\sqrt{p^+} \theta)$. This is not a satisfactory kinetic term because we also would need a term with $p^+ \theta \gamma_+ \partial_\sigma \theta$. Such a term would be obtained if the action contains a term of the form $(\partial_t x^+) \theta \gamma_+ \partial_\sigma \theta$, or in covariant notation $\epsilon^{\mu\nu} (\partial_\mu x^m) \theta \gamma_m \partial_\nu \theta$. The extra kinetic term $\epsilon^{\mu\nu} \partial_\mu x^m \theta \gamma_m \partial_\nu \theta$ is part of a Wess-Zumino term, (see appendix B).

Rigid susy (1.1) and $\delta_\epsilon (\partial_\sigma x^+) = 0$ would lead to $\epsilon \gamma^+ \partial_\sigma \theta = 0$. This suggests that the light-cone gauge for θ should read $\gamma^+ \theta = 0$. Since $\gamma^+ \theta = 0$, also $\theta \gamma^+ = 0$, and using $\{\gamma^+, \gamma^-\} = 1$, one would also find that $\theta \gamma^I \partial_\sigma \theta = 0$ for $I = 1, \dots, 8$. So, then we would find in the light cone gauge that the action for θ becomes a free action, a good starting point for string theory at the quantum level.

In order that these steps are correct, we would need a local fermionic symmetry which would justify the gauge $\gamma^+ \theta = 0$. Pursuing this line of thought, one arrives then at the crucial question: does the sum of the supersymmetric line element and the WZNW term contain a new fermionic symmetry with half as many parameters as there are θ components? The answer is affirmative, and the κ -symmetry is briefly discussed at the end of appendix

B, but since we shall not need the explicit form of the κ symmetry transformation laws, we do not give them.

The superstring action is very complicated already in a flat background. We extract from it a set of first class constraints $d_{z\alpha} = 0$, from which we build the BRST charge, and at all stages we work with a free action. The precise way to obtain $d_{z\alpha}$ from the classical superstring action is discussed in appendix B.

3. Determining the theory from the nilpotency the BRST charge

We now start our program of determining the theory the BRST charge and the ghost content) by requiring nilpotency of the BRST transformations. We consider only θ for simplicity (we have also extended our work to two θ 's. We shall be careful (for once) with aspects such as reality and normalizations. The BRST transformations preserve reality and are generated by ΛQ where Λ is imaginary and anti-commuting. It then follows that Q should also be antihermitian in order that ΛQ be antihermitian. For any field, we define the s transformations as BRST transformations without Λ , so $\delta_B \Phi = [\Lambda Q, \Phi]$ and $s\Phi = [Q, \Phi]_{\pm}$. The s -transformations have reality properties which follows from the BRST transformations (which preserve reality).

We begin with

$$Q = \int i\lambda^\alpha d_{z\alpha}, \quad (3.1)$$

where $d_{z\alpha}$ is given in Appendix C and $\int = \frac{1}{2\pi} \oint dz$, which is indeed antihermitian because $d_{z\alpha}$ is antihermitian. (We have performed a Wick rotation in appendix C, in order to be able to use the conventional tools of conformal field theory, but the reality properties hold in Minkowski space). The BRST operator depends on Heisenberg fields which satisfy the field equations, and since we work with a free action, $\bar{\partial}\lambda^\alpha = 0$ and $\bar{\partial}d_{z\alpha} = 0$ so that in flat space $\lambda^\alpha d_{z\alpha}$ is a holomorphic current, namely $\bar{\partial}(\lambda^\alpha d_{z\alpha}) = 0$.

The field $d_{z\alpha}$ contains a term $p_{z\alpha}$, where $p_{z\alpha}$ is the momentum conjugate to θ^α and it is antihermitian since $p_{z\alpha}$ is antihermitian as can be seen from the action $\int d^2z p_{z\alpha} \bar{\partial}\theta$. The factor $\frac{1}{2}$ in $d_{z\alpha}$ is fixed by requiring that the OPE⁴ of d_α with d_β be proportional to Π_w^m . The expression for Π_z^m is real and fixed by spacetime susy.

⁴ The OPE of d_α with d_β is evaluated using $\partial x^m(z) \partial x^n(w) \sim -\eta^{mn}(z-w)^{-2}$ and $p_{z\alpha}(z) \theta^\beta(w) \sim \delta_\alpha^\beta (z-w)^{-1}$.

The operators $d_{z\alpha}$ generate a closed algebra of current with a central charge

$$\begin{aligned} d_\alpha(z)d_\beta(w) &\sim 2i\frac{\gamma_{\alpha\beta}^m\Pi_m(w)}{z-w}, & d_\alpha(z)\Pi^m(w) &\sim -2i\frac{\gamma_{\alpha\beta}^m\partial\theta^\beta(w)}{z-w}, \\ \Pi^m(z)\Pi^n(w) &\sim -\frac{1}{(z-w)^2}\eta^{mn}, & d_\alpha(z)\theta^\beta(w) &\sim \frac{1}{z-w}\delta_\alpha^\beta. \end{aligned} \quad (3.2)$$

Acting with (3.1) on θ^α , one obtains $s\theta^\alpha = i\lambda^\alpha$, and acting on λ^α yields $s\lambda^\alpha = 0$. Nilpotency on θ^α and λ^α is achieved. Repeating this procedure on x^m gives $sx^m = \lambda\gamma^m\theta$, but since $s^2x^m = i\lambda\gamma^m\lambda$ does not vanish, we introduce a new ghost ξ^m by setting $sx^m = \lambda\gamma^m\theta + \xi^m$ and choosing the BRST transformation law of ξ^m such that the nilpotency on x^m is obtained. This leads to $s\xi^m = -i\lambda\gamma^m\lambda$. Nilpotency on x^m is now achieved, but s has acquired an extra term⁵ $Q' = -\int \xi^m\Pi_{zm}$ where we recall $\Pi_z^m = \partial_z x^m - i\theta\gamma^m\partial_z\theta$. Nilpotency on $p_{z\alpha}$, or equivalently on $d_{z\alpha}$, is obtained by further modifying the sum of $Qd_{z\alpha} = -2\Pi_z^m(\gamma_m\lambda)_\alpha$ and $Q'd_{z\alpha} = -2i\xi^m(\gamma_m\partial_z\theta)_\alpha$ by adding $Q''d_{z\alpha} = \partial_z\chi_\alpha$ and fixing the BRST law of χ_α such that nilpotency on $d_{z\alpha}$ is achieved.⁶ This yields $Q\chi_\alpha = 2\xi^m(\gamma_m\lambda)_\alpha$ and $Q^2\chi_\alpha = 0$ due to a Fierz rearrangement involving three chiral spinors. At this point we have achieved nilpotency on $\theta^\alpha, x^m, d_{z\alpha}$ and $\lambda^\alpha, \xi^m, \chi_\alpha$. We introduce the antighosts $w_{z\alpha}, \beta_{zm}, \kappa_z^\alpha$ for the ghosts $\lambda^\alpha, \xi^m, \chi_\alpha$ and find that $s\Phi = [Q, \Phi]$ with

$$Q = \oint \left(i\lambda^\alpha d_{z\alpha} - \xi^m \Pi_{zm} - \chi_\alpha \partial_z \theta^\alpha - 2\xi^m (\kappa_z \gamma_m \lambda) - i\beta_{zm} \lambda \gamma^m \lambda \right) \quad (3.3)$$

reproduces all BRST laws obtained so far.

Unfortunately, the BRST charge (3.3) fails to be nilpotent and therefore the concept of the BRST cohomology is at this point meaningless. In order to repair this problem, we could proceed in two different ways: *i*) either continuing with our program and requiring nilpotency on each field separately (on the antighosts $\beta_{zm}, \kappa_z^\alpha$ and w_α); or *ii*) terminate this process by hand in one stroke by adding a ghost pair (b, c_z) as we now explain. We begin with

$$Q^2 = \int A_z, \quad A_z = \xi_m \partial_z \xi^m + i\lambda^\alpha \partial_z \chi_\alpha - i\chi_\alpha \partial_z \lambda^\alpha. \quad (3.4)$$

The non-closure term A_z is due to the double poles in (3.2). By direct computation we establish that the anomaly $\int A_z$ is BRST invariant, as it should be according to consistency, $[Q, A_z] = \partial_z Y$ where $Y = \frac{i}{2}\xi_m \lambda \gamma^m \lambda$. If we define

$$Q' = Q + \int \left(c_z - \frac{1}{2} b B_z \right), \quad (3.5)$$

⁵ Spacetime susy requires that Q' depends on Π_z^m instead of, for example $\partial_z x^m$.

⁶ Since $(Q + Q')d_{z\alpha} = \partial_z(-2\xi^m \gamma_m \lambda)_\alpha$, we add a term $\partial_z \chi_\alpha$ instead of a field $\chi_{z\alpha}$.

with an hermitian c_z and an antihermitian b , we find that

$$Q'^2 = \int \left((A_z - B_z) + \frac{1}{2}b[Q, B_z] \right), \quad (3.6)$$

and, requiring that Q' be nilpotent, a solution for B_z is obtained by imposing

$$[Q, B_z] = 0, \quad B_z = A_z + \partial X, \quad [Q, X] = -Y \quad (3.7)$$

which is satisfied by $X = -\frac{i}{2}\chi_\alpha\lambda^\alpha$. Then one gets

$$B_z = \xi_m \partial_z \xi^m + \frac{1}{2}\lambda^\alpha \partial_z \chi_\alpha - \frac{3}{2}\chi_\alpha \partial_z \lambda^\alpha. \quad (3.8)$$

However, any Q' of the form $\int c_z + \text{“more”}$ can be always brought in the form $\int c_z$ by a similarity transformations choosing the term denoted by “more” appropriately, namely as follows

$$\begin{aligned} Q' &= \left[e^{\frac{1}{2} \int (-R_z - b S_z - b \partial_z b T)} \int c_z e^{\frac{1}{2} \int (R_z + b S_z + b \partial_z b T)} \right] \\ &= \int (c_z + S_z - b \partial_z T) + \left[\int (S_z - b \partial_z T), \mathcal{U} \right] + \frac{1}{2} \left[\left[\int (S_z - b \partial_z T), \mathcal{U} \right], \mathcal{U} \right] + \dots \end{aligned} \quad (3.9)$$

where $\mathcal{U} = \int (R_z + b S_z + b \partial_z b T)$. The R_z, S_z and T are hermitian polynomials in all fields except c_z, b with ghost numbers 0, 1, 2, respectively. The solution in (3.5) and (3.8) corresponds to a particular choice of R_z, S_z and T , but any other choice also yields a nilpotent BRST charge. The operator $Q' = e^{-\mathcal{U}} \int c_z e^{\mathcal{U}}$ has trivial cohomology in the space of local vertex operators with vanishing conformal spin, because any $\mathcal{O}(w)$ satisfying $\int c_z \mathcal{O}(w) = 0$ can always be written as $\mathcal{O}(w) = b_0 \mathcal{G}(w)$ where $\mathcal{G}(w) = \int c_z \mathcal{O}(w)$. (Note that $\mathcal{O}(w)$ cannot depend on c_z , and $c_0 = \int c_z$).

We shall restrict the space of vertex operators in which Q acts, in order to obtain non-trivial cohomology. We achieve this by introducing a new quantum number, called grading, and requiring that vertex operators have non-negative grading. In the smaller space of non-negative grading the similarity transformation cannot transform each Q into the form $\int c_z$, and we shall indeed obtain non-trivial cohomology, namely the correct cohomology.

We have at this point obtained a nilpotent BRST charge, and a set of ghost (and antighost) fields (whose geometrical meaning at this point is becoming clear). It is time to revert to the issue of the central charge. Since all fields are free fields, one simply needs to add the central charge of each canonical pair: $c = 20$. So the central charge does not

vanish, and to remedy this obstruction, we add by hand an anticommuting vector pair (ω^m, η_z^m) which contributes -2×10 to c . The BRST charge does not contain ω^m and η_z^m , hence ω^m and η_z^m are BRST inert.

The reader (and the authors) may feel uncomfortable with these rescue missions by hand, a good theory should produce all fields automatically without outside help. Fortunately, we can announce that a more fundamental way of proceeding, by continuing to require nilpotency on the antighosts and then on the new fields which are introduced in this process, produces the pair (ω^m, η_z^m) ! We are in the process of writing these consideration up, and hopefully also the pair (b, c_z) will be automatically produced in this way.

Our results obtained by elementary methods and ad hoc addition, display nevertheless a few striking regularities, which confirm us in our belief that we are on the right track.

4. The notion of the grading

In our work we define physical states by means of vertex operators which satisfy two conditions

- i) They are in the BRST cohomology
- ii) They should have non-negative grading [7].

The grading is a quantum number which was initially obtained from the algebra of the abstract currents $d_{z\alpha}, \Pi_z^m$ and $\partial_z \theta^\alpha$. Assigning grading -1 to $d_{z\alpha}$, we assign grading $+1$ to the corresponding ghost λ^α . We then require that the grading be preserved in the operator product expansion. From $dd \sim \Pi$ we deduce that Π_z^m has grading -2 , so ξ^m has grading 2 . Then $d\Pi \sim \partial\theta$ assigns grading -3 to $\partial\theta$, and thus grading $+3$ to χ . The grading of the ghosts b and c is more subtle, but it can be obtained in the same spirit. From $d\partial\theta \sim (z-w)^{-2}$ and $\text{III} \sim (z-w)^{-2}$ we introduce a central charge generator I which has grading -4 . The corresponding ghost c_z has grading 4 . All antighosts have opposite grading from the ghosts. The trivial ghost pair ω^m, η_z^m has grading $(4, -4)$ because it is part of a quartet of which the grading of the other members is already known [7]. With these grading assignments to the ghost fields, the BRST charge can be decomposed into pieces of non-negative grading $Q = \sum_{n=0}^4 Q_n$ and it maps the subspace of the Hilbert space with non-negative grading into itself. In [8], the equivalence with Berkovits' pure spinor formulation has been proven.

According to the grading condition ii), the most general expression for the massless

vertex in the case of open superstring is given by

$$\mathcal{O} = \lambda^\alpha A_\alpha + \xi^m A_m + \chi_\alpha W^\alpha + b - \text{terms} \quad (4.1)$$

where A_α, A_m and W^α are arbitrary superfields, so $A_\alpha = A_\alpha(x, \theta)$, etc.. Requiring non-negative grading, the following combinations

$$b\lambda^\alpha\lambda^\beta, \quad b\lambda^\alpha\xi^m, \quad (4.2)$$

are not allowed. Finally, requiring the BRST invariance, one easily derives the equations of motion for $N = 1$ SYM in $D = (9, 1)$. Along the same lines, one can study the closed string or massive operators and one finds the complete correct spectrum of the open or closed superstring.

The notion that one must restrict the space of the vertex operators is not new by itself: in the spinning (RNS) string, one should restrict the commuting susy ghosts to non-negative mode numbers [9], and also in the bosonic string one has the condition that vertex operators are annihilated by b_0 (where b_0 belongs to b_{zz}). We have recently shown that the concept of grading is nothing else than the “pure ghost number” of homological perturbation theory [10]. So there is, after all, a deeper geometrical meaning to the ideas we have developed.

5. Our program applied to Yang-Mills theory

The program of determining the theory by starting from a suitable set of constraints $d_{z\alpha}$ and a free action (for x^m, θ^α and d_α) leads to a nilpotent BRST charge and a free action in the case of the superstring. Since the ideas are new we would like to see them at work in a simpler example. We therefore study in this section whether also for standard pure Yang-Mills field theory similar ideas can be implemented and what results they lead to.

We begin with Yang-Mills fields and write the gauge transformations as BRST-like transformations by introducing a ghost field c^a for each infinitesimal gauge parameter

$$s_0 A = \nabla c, \quad s_0 c = 0. \quad (5.1)$$

The law $s_0 c = 0$ corresponds to $s_0 \lambda^\alpha$ and $s_0 A = \nabla c$ corresponds to $s_0 \theta^\alpha = i\lambda^\alpha$. In string theory we have “brackets” which are the contraction and propagators of conformal field

theory. To also introduce brackets for A and c , we introduce antifields A^* and c^* and define the antibracket

$$(X, Y) = \frac{\delta_l X}{\delta z^A} \frac{\delta_r Y}{\delta z_A^*} - \frac{\delta_l X}{\delta z_A^*} \frac{\delta_r Y}{\delta z^A} \quad (5.2)$$

for any X, Y in the algebra \mathcal{A} to construct the rest of the terms in s . We introduce A^*, c^* the conjugate variable to A, c such that $(A_\mu^a(x), A_{b\nu}^*(y)) = \delta_b^a \eta_{\mu\nu} \delta^4(x - y)$ and $(c^a(x), c_b^*(y)) = \delta_b^a \delta^4(x - y)$. Notice that though the fields A^*, c^* are antifields themselves, in the present section we assign antifield number zero to them. In addition, we are not taking into account the Yang-Mills equation of motion, but we are only discussing the gauge invariant observables and not the observables modulo equations of motion. In the following, we will use the antifields as conjugate momenta. The relation between antibracket (5.2) and Poisson bracket has been extensively discussed in the literature and we refer to [11] and [12].

The transformation laws in (5.1) are generated by $S_0 = -\int A^* \nabla c$. This corresponds to $Q_0 = \int i \lambda^\alpha d_\alpha$. The symmetry in (5.1) is not the BRST symmetry because it is not nilpotent $s_0^2 A = -\frac{1}{2} \nabla [c, c]$ where $[\cdot, \cdot]$ is the Lie algebra bracket. However, we can apply again the ideas of homological perturbation theory to impose $[c, c] = 0$ as a constraint. This resembles the pure spinor constraint (1.4). The constraint $[c, c] = 0$ is an abelian first class constraint and it generates the gauge transformations $\Delta_\epsilon c^* = (\frac{1}{2} \epsilon \cdot [c, c], c^*) = [\epsilon, c]$ where ϵ is a vector in the adjoint representation and \cdot is the trace operation. Finally, the square of the s_0 transformations of the fields gives (with $\nabla c = d c - [A, c]$)

$$s_0^2 A = \nabla \left(-\frac{1}{2} [c, c] \right), \quad s_0^2 A^* = \left[-\frac{1}{2} [c, c], A^* \right], \quad s_0^2 c = 0, \quad s_0^2 c^* = \Delta_{\nabla A^*} A^*, \quad (5.3)$$

which shows that s_0 is nilpotent on the surface of the constraints modulo gauge transformations. We introduce a new anticommuting field η^* and a differential δ such that δ maps η^* into the constraint, and δ has antifield number $\text{af}(\delta) = -1$, and $\text{af}(A^*) = \text{af}(c^*) = 0$.

$$\delta \eta^* = -\frac{1}{2} [c, c], \quad \delta A = 0, \quad \delta c = 0, \quad (5.4)$$

$$\text{af}(\eta^*) = 1, \quad \text{af}(A) = 0, \quad \text{af}(c) = 0.$$

We then define the pure ghost number pg as the sum of the antifield number and the ghost number. It is easy to check that $\text{pg}(\eta^*) = 0$. Applying the theorem of HPT, the two operations can be merged in only one nilpotent $s = \delta + s_0 + \dots$ since the BRST-like transformation s_0 is nilpotent modulo δ -exact terms.

A simple exercise shows that

$$\begin{aligned}
\delta \eta^* &= -\frac{1}{2}[c, c], \quad s_0 \eta^* = -[\eta^*, c], \quad s_1 \eta^* = -\frac{1}{2}[\eta^*, \eta^*], \\
\delta \eta &= 0, \quad s_0 \eta = -[\eta, c], \quad s_1 \eta = [\eta, \eta^*] - \nabla A^*, \\
\delta A^* &= 0, \quad s_0 A^* = [c, A^*], \quad s_1 A^* = [\eta^*, A^*], \\
\delta A &= 0, \quad s_0 A = \nabla c, \quad s_1 A = \nabla \eta^*, \\
\delta c^* &= [c, \eta], \quad s_0 c^* = -\nabla A^* + [\eta^*, \eta], \quad s_1 c^* = 0, \\
\delta c &= 0, \quad s_0 c = 0, \quad s_1 c = 0.
\end{aligned} \tag{5.5}$$

As we already recalled, the construction of the BRST charge, which contains both the Koszul-Tate differential δ and the BRST-like differential s_0 , is unique up to a (anti) canonical⁷ transformation, for example a field redefinition. If we shift η^* with the ghost field and we rename this field C (and in the same way $\eta \equiv C^*$), we find out that these transformations can be generated by $sX = (S_{\text{af}}, X)$ where S_{af} is

$$S_{\text{af}} = \int d^4x \left(A^* \nabla C + \frac{1}{2} C^* [C, C] \right). \tag{5.6}$$

The Lagrangian S_{af} is clearly the usual antifield dependent terms of the Yang-Mills Lagrangian. Finally, one can study the cohomology of the BRST operator s and one easily finds out that the cohomology coincides with the gauge invariant observables of YM theory.

Notice that by means of the redefinition, we cannot use the antifield number to select the resolution of the Koszul-Tate δ any longer. Fortunately, in the present case it is easy to study the cohomology $H(s)$ directly. In addition, the antifield number is protected (it cannot be too negative!!) because it is equal to the ghost number.

6. Acknowledgments

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⁷ Anticanonical transformations are generated by the antibracket $\phi \rightarrow \phi' = \phi + (\mathcal{F}, \phi)$ where \mathcal{F} is a fermionic generator.

7. Appendix A: Majorana and Weyl spinors in $D = (9, 1)$.

In $D = (9, 1)$ dimensions, we use ten real $D = (9, 1)$ Dirac-matrices $\Gamma^m = \{I \otimes (i\tau_2), \sigma^\mu \otimes \tau_1, \chi \otimes \tau_1\}$ where $m = 0, \dots, 9$ and $\mu = 1, \dots, 8$. The σ^μ are eight real symmetric 16×16 off-diagonal Dirac matrices for $D = (8, 0)$, while χ is the real 16×16 diagonal chirality matrix in $D = 8^8$. So $\chi = \sigma_1, \dots, \sigma_8$, $\chi^T = \chi$ and $\chi^2 = 1$. The chirality matrix in $D = (9, 1)$ is then $I \otimes \tau_3$ and the $D = (9, 1)$ charge conjugation matrix C , satisfying $C\Gamma^m = -\Gamma^{m,T}C$, is numerically equal to $C = \Gamma^0$. If one uses spinors $\Psi^T = (\lambda_L, \zeta_R)$ with spinor indices λ_L^α and $\zeta_{R,\dot{\beta}}$, the index structure of the Dirac matrices, the charge conjugation matrix C , satisfying $C\Gamma^m = -\Gamma^{m,T}C$ is numerically equal to Γ^0 , and the chirality matrix $\Gamma_\# \equiv \Gamma^0\Gamma^1 \dots \Gamma^9 = I_{16 \times 16} \otimes \sigma_3$ is as follows

$$\Gamma^m = \begin{pmatrix} 0 & (\sigma^m)^{\alpha\dot{\beta}} \\ (\tilde{\sigma}^m)_{\dot{\beta}\gamma} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c_\alpha^{\dot{\beta}} \\ c_\gamma^{\dot{\beta}} & 0 \end{pmatrix}, \quad \Gamma_\# = \begin{pmatrix} I_{16 \times 16} & 0 \\ 0 & -I_{16 \times 16} \end{pmatrix},$$

where $\sigma^m = \{I, \sigma^\mu, \chi\}$ and $\tilde{\sigma}^m = \{-I, \sigma^\mu, \chi\}$. The matrices $c_\alpha^{\dot{\beta}}$ and $c_\gamma^{\dot{\beta}}$ are numerically equal to $I_{16 \times 16}$ and $-I_{16 \times 16}$, respectively. Thus the λ^α are chiral and the $\zeta_{\dot{\beta}}$ are antichiral. This explains the spinorial index structure of the Γ^m . The matrices γ_m satisfy $\gamma_{\alpha\beta}^m \gamma^{n\beta\gamma} + \gamma_{\alpha\beta}^n \gamma^{m\beta\gamma} = 2\eta^{mn}\delta_\alpha^\gamma$ and $\gamma_{m(\alpha\beta} \gamma_{\gamma)\delta}^m = 0$. The latter relation makes Fierz rearrangements very easy.

In applications we need the matrices $C\Gamma^m$ (for example in (1.4)). Direct matrix multiplication shows that $C\Gamma^m$ is given by

$$\Gamma^m = \begin{pmatrix} (\tilde{\sigma}^m)_{\alpha\beta} & 0 \\ 0 & -(\sigma^m)^{\dot{\beta}\dot{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} \gamma_{\alpha\beta}^m & 0 \\ 0 & (\gamma^m)^{\beta\alpha} \end{pmatrix}. \quad (7.1)$$

We only use the real 16×16 symmetric matrices $\gamma_{\alpha\beta}^m = \tilde{\sigma}_{\alpha\beta}^m$ and $\gamma_m^{\dot{\alpha}\dot{\beta}} = -\sigma^{m\dot{\alpha}\dot{\beta}}$ in the text, and we omit the dots for reasons we now explain.

⁸ The 8 real 16×16 matrices of $D = (8, 0)$ can be obtained from a set of 7 pure imaginary 8×8 matrices λ^i for $D = (7, 0)$ as follows $\sigma^\mu = \{\lambda^i \otimes \sigma_2, I_{8 \times 8} \otimes \sigma_1\}$. The seven 8×8 matrices λ^i themselves can be obtained from the representation $\gamma^k = \sigma^k \otimes \tau^2$, $\gamma^4 = 1 \otimes \tau^1$, and $\gamma^5 = 1 \otimes \tau^3$ for $D = (3, 1)$ with real symmetric matrices $\gamma^2, \gamma^4, \gamma^5$ and imaginary antisymmetric γ^1, γ^3 as follows

$$\lambda^i = \{\gamma^2 \otimes \sigma_2, \gamma^4 \otimes \sigma_2 \otimes \gamma^5 \otimes \sigma_2, \gamma^1 \otimes 1, \gamma^3 \otimes 1, i\gamma^2\gamma^4\gamma^5 \otimes \sigma_1, i\gamma^2\gamma^4\gamma^5 \otimes \sigma_3\}.$$

The Lorentz generators are given by

$$L^{mn} = \frac{1}{4}(\Gamma^m \Gamma^n - \Gamma^n \Gamma^m) = \begin{pmatrix} \frac{1}{2}\sigma^{m,\alpha\dot{\beta}}\tilde{\sigma}_{\dot{\beta}\gamma}^n - m \leftrightarrow n & 0 \\ 0 & -\frac{1}{2}\sigma^{m,\alpha\dot{\beta}}\tilde{\sigma}_{\dot{\beta}\gamma}^n - m \leftrightarrow n \end{pmatrix} \quad (7.2)$$

Hence the chiral spinors λ^α and the antichiral $\zeta_{\dot{\beta}}$ form separate representation for $SO(9, 1)$. These representations are inequivalent because σ^m and $\tilde{\sigma}^m$ are equal except for $m = 0$ where $\sigma^0 = I$ but $\tilde{\sigma}^0 = -I$, and there is no matrix S satisfying $S\sigma^\mu = -\sigma^\mu S$ and $S\chi = -\chi S$. (From $S\sigma^\mu = -\sigma^\mu S$ it follows that $S\chi = +\chi S$). We denote these real inequivalent representation by 16 and 16', respectively.

In $D = (9, 1)$ dimensions one cannot raise or lower spinor indices with the charge conjugation matrix, because C is off-diagonal. In $D = (3, 1)$, on the other hand, C is diagonal and is given by $C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\beta}\dot{\alpha}} \end{pmatrix}$, and therefore one can raise and lower the indices with the charge conjugation matrices $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. For that reason one has in $D = (3, 1)$ four kinds of spinors $\lambda^\alpha, \lambda_\alpha, \chi_{\dot{\beta}}$ and $\chi^{\dot{\beta}}$. In $D = (9, 1)$ dimension one has only spinors λ^α and $\chi_{\dot{\beta}}$, and thus one may omit the dots on $\chi_{\dot{\beta}}$ without causing confusion.

We conclude that chiral spinors are given by λ^α , antichiral spinors by χ_α and in the text we use the twenty real symmetric 16×16 matrices $\gamma_{\alpha\beta}^m$ and $\gamma^{m,\alpha\beta}$ (omitting again the dots in the latter). The usual Fierz rearrangement for 3 chiral spinors becomes then simply the statement that $\gamma_{\alpha\beta}^m \gamma_{m\gamma\delta}$ vanishes when totally symmetrized in the indices α, β and γ .

8. Appendix B: The WZNW term.

We follow [13]. The WZNW term \mathcal{L}_{WZ} is proportional to $\epsilon^{\alpha\beta}$ (with $\alpha, \beta = 0, 1$) hence $\mathcal{L}_{WZ} d^2x$ can be written as a 2-form

$$\omega_2 \equiv \mathcal{L}_{WZ} d^2x. \quad (8.1)$$

Since ω_2 is susy invariant up to a total derivative, we have

$$\delta_\epsilon \omega_2 = dX. \quad (8.2)$$

Define now a 3-form ω_3 as follows: $\omega_3 = d\omega_2$. Then clearly,

$$\delta_\epsilon \omega_3 = 0, \quad d\omega_3 = 0. \quad (8.3)$$

From $\delta_\epsilon \omega_3 = 0$ it is natural to try to construct ω_3 from the susy-invariant 1-forms $\Pi^m = dx^m - i \sum_j \theta^j \gamma^m d\theta_j$ and $d\theta^i$. Lorentz invariance then yields only one possibility

$$\omega_3 = a_{ij} \Pi^m d\theta^i \gamma^m d\theta^j . \quad (8.4)$$

where a_{ij} is a real symmetric $N \times N$ matrix. We diagonalize a_{ij} by a real orthogonal transformation (which leaves Π^m , and thus \mathcal{L}_1 , invariant). Then $d\omega_3 = -i (\sum_i d\theta^i \gamma^m d\theta^i) (\sum_k a_k d\theta^k \gamma^m d\theta^k)$. In $d\omega_3$ the direct terms cancel due to the standard identity $\gamma^m d\theta^1 (d\theta^1 \gamma_m d\theta^1) = 0$, while the cross-terms cancel only if $N = 2$ and if the diagonal matrix a_{ij} has entries $(+1, -1)$. Hence

$$\omega_3 = -i \Pi^m (d\theta^1 \gamma_m d\theta^1 - d\theta^2 \gamma^m d\theta^2) . \quad (8.5)$$

Using that $\omega_3 = d\omega_2$, we find the WZNW term up to an overall constant

$$\mathcal{L}_{WZ} = -\frac{1}{\pi} \epsilon^{\mu\nu} [i \partial_\mu x^m (\theta^1 \gamma_m \partial_\nu \theta^1 - \theta^2 \gamma_m \partial_\nu \theta^2) + \theta^1 \gamma_m \partial_\mu \theta^1 \theta^2 \gamma^m \partial_\nu \theta^2] . \quad (8.6)$$

Indeed,

$$d(\mathcal{L}_{WZ} d^2 x) \sim -i dx^m (d\theta^1 \gamma_m d\theta^1 - d\theta^2 \gamma^m d\theta^2) + (\theta^1 \gamma_m d\theta^1 d\theta^2 \gamma_m d\theta^2 - d\theta^1 \gamma_m d\theta^1 \theta^2 \gamma^m d\theta^2) \quad (8.7)$$

which is equal to

$$\omega_3 = -i (dx^m - \theta^1 \gamma_m d\theta^1 - \theta^2 \gamma^m d\theta^2) (d\theta^1 \gamma_m d\theta^1 - d\theta^2 \gamma_m d\theta^2) . \quad (8.8)$$

Note that the WZNW term is antisymmetric in θ^1 and θ^2 while \mathcal{L}_1 is symmetric. Only the sum of \mathcal{L}_1 and \mathcal{L}_{WZ} is κ -invariant, up to a total derivative. The κ -transformation rules for x^m read $\delta_\kappa x^m = -\epsilon^i \gamma^m \delta_\kappa \theta^i$ with the opposite sign to the susy rule. The expression for $\delta_\kappa \theta^\alpha$ and $\delta_\kappa \sqrt{-h} h^{\mu\nu}$ are complicated, involving self-dual and antiselfdual anticommuting gauge parameters with 3 indices, but we do not need them. We begin with the BRST law $s \theta^\alpha = i \lambda^\alpha$ where λ^α is an unconstrained ghost field, but the precise classical action to which this corresponds is not known at the present. That does not matter as long as we can construct the complete quantum theory, although knowledge of the classical action might clarify the results obtained at the quantum level.

For the open string one has the following boundary conditions at $\sigma = 0, \pi$

$$\theta^{1i} = \theta^{2i} , \quad \epsilon^{1i} = \epsilon^{2i} , \quad h^{\sigma\beta} \partial_\beta x^m = 0 , \quad \kappa_t^{1i} = k_t^{2i} . \quad (8.9)$$

9. Appendix C: A useful identity for the superstring

The superstring action is given by

$$\mathcal{L} = -\frac{1}{2\pi} \eta_{mn} \Pi_\mu^m \Pi^{n\mu} - \mathcal{L}_{WZ}, \quad (9.1)$$

$$\mathcal{L}_{WZ} = -\frac{1}{\pi} \epsilon^{\mu\nu} [i\partial_\mu x^m (\theta^1 \gamma_m \partial_\nu \theta^1 - \theta^2 \gamma_m \partial_\nu \theta^2) + \theta^1 \gamma_m \partial_\mu \theta^1 \theta^2 \gamma^m \partial_\nu \theta^2]$$

where Π_μ^m is given in (1.2). For definiteness we choose $\epsilon^{01} = 1$ and $\eta^{\mu\nu}$ as well as η^{mn} have $\eta^{00} = -1$. This action is real.

By just writing out all the term, the action can be re-written with chiral derivatives

$$\begin{aligned} -\pi \mathcal{L} = & \eta_{mn} \partial x^m \bar{\partial} x^n - \partial x^m \theta^1 \gamma_m \bar{\partial} \theta^1 - \bar{\partial} x^m \theta^2 \gamma_m \partial \theta^2 \\ & + \frac{1}{2} (\theta^1 \gamma^m \bar{\partial} \theta^1) (\theta^1 \gamma_m \partial \theta^1 + \theta^2 \gamma_m \partial \theta^2) + \frac{1}{2} (\theta^2 \gamma^m \partial \theta^2) (\theta^1 \gamma_m \bar{\partial} \theta^1 + \theta^2 \gamma_m \bar{\partial} \theta^2) \end{aligned} \quad (9.2)$$

with $\partial = \partial_\sigma - \partial_t$ and $\bar{\partial} = \partial_\sigma + \partial_t$.

Except for the purely bosonic terms, all terms involve either $\partial\theta$ or $\bar{\partial}\theta$. Hence we can write the action as

$$-\pi \mathcal{L} = \eta_{mn} \partial x^m \bar{\partial} x^n + (p_{1\alpha})_{\text{Sol}} \bar{\partial} \theta^{1\alpha} + (p_{2\alpha})_{\text{Sol}} \partial \theta^{2\alpha} \quad (9.3)$$

where $(p_{i\alpha})_{\text{Sol}}$ are complicated composite expressions. We restrict ourselves to the left-moving sector, setting $\theta^2 = p_2 = 0$.

We can then also write the action with independent $p_{i\alpha}$ if we impose the constraint that $d_{i\alpha} \equiv p_{i\alpha} - (p_{i\alpha})_{\text{Sol}}$ vanishes. Finally, the complete expressions are given by

$$d_{1\alpha} = p_{1\alpha} + \partial x^m \theta^1 \gamma_m \bar{\partial} \theta^1 - \frac{1}{2} (\theta^1 \gamma^m \bar{\partial} \theta^1) (\theta^1 \gamma_m \partial \theta^1 + \theta^2 \gamma_m \partial \theta^2), \quad (9.4)$$

$$d_{2\alpha} = p_{2\alpha} + \partial x^m \theta^2 \gamma_m \bar{\partial} \theta^2 - \frac{1}{2} (\theta^2 \gamma^m \bar{\partial} \theta^2) (\theta^1 \gamma_m \partial \theta^1 + \theta^2 \gamma_m \partial \theta^2).$$

In the text we work with the free action with independent fields $p_{i\alpha}$. The $d_{i\alpha}$ are transferred to the BRST charge where they are multiplied by the independent unconstrained real chiral commuting spinors λ^α . To make use of the calculation technique of conformal field theory, we made a Wick rotation $t \rightarrow -i\tau$, $\partial_t \rightarrow +i\partial_\tau$ and $\partial = \partial_\sigma - \partial_t \rightarrow \partial = \partial_\sigma - i\partial_\tau$ and analogously for $\bar{\partial}$. We also restrict ourselves to only one sector with $\theta = \theta^1$ and $d_\alpha = d_{1\alpha}$, by setting $\theta^2 = 0$. For a treatment which describes both sectors, we refer to [14].

10. Appendix D: Solution of the pure spinor constraints.

In this appendix we discuss a solution of the constraint that the chiral spinors λ are pure spinors. The equation to be solved reads

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0, \quad (10.1)$$

where λ^α are complex chiral (16-component) spinors. We shall decompose λ w.r.t. a non-compact version of the $SU(5)$ subgroup of $SO(9,1)$ as $|\lambda\rangle = \lambda_+|0\rangle + \frac{1}{2!}\lambda_{ij}a^i a^j|0\rangle + \frac{1}{4!}\lambda_{jklm}a^j a^k a^l a^m|0\rangle$. This decomposition corresponds to $\underline{16} = \underline{1} + \underline{10} + \underline{5}^*$. Then we shall show that the constraints express the $\underline{5}^*$ in terms of the $\underline{1}$ and $\underline{10}$. Hence there are 11 independent complex components in λ . We shall prove that λ is complex and not a Majorana spinor, so $\bar{\lambda}_D \equiv \lambda^\dagger i\gamma^0$ differs from $\bar{\lambda}_M = \lambda^T C$. (Recall that a Majorana spinor is defined by the condition $\bar{\lambda}_D = \bar{\lambda}_M$).

The Dirac matrices in $D = (9,1)$ dimensions satisfy $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$, where η^{mn} is diagonal with entries $(-1, +1, \dots, 1)$ for $m, n = 0, \dots, 9$. We combine them into 5 annihilation operators a_j and 5 creation operators $a^j = a_j^\dagger$ as follows

$$a_1 = \frac{1}{2}(\Gamma^1 + i\Gamma^2), \quad a_2 = \frac{1}{2}(\Gamma^3 + i\Gamma^4), \quad \dots \quad a_5 = \frac{1}{2}(\Gamma^9 - \Gamma^0). \quad (10.2)$$

$$a^1 = \frac{1}{2}(\Gamma^1 - i\Gamma^2), \quad a^2 = \frac{1}{2}(\Gamma^3 - i\Gamma^4), \quad \dots \quad a^5 = \frac{1}{2}(\Gamma^9 + \Gamma^0).$$

Clearly $\{a_i, a^j\} = \delta_i^j$ for $i, j = 1, \dots, 5$. We introduce a vacuum $|0\rangle$ with $a_i|0\rangle = 0$. By acting with one or more a^j on $|0\rangle$, we obtain 32 states $|A\rangle$ with $A = 1, \dots, 32$. Similarly, we introduce a state $\langle 0|$ which satisfies $\langle 0|a^j = 0$ and we create 32 states $\langle B|$ by acting with one or more a_i on $\langle 0|$. We choose the states $\langle B|$ as $|A\rangle^\dagger$. For example, if $|A\rangle = a^{i_1} \dots a^{i_k}|0\rangle$ then $\langle A| = \langle 0|a_{i_k} \dots a_{i_1}$. Then $\langle A|B\rangle = \delta^A_B$.

Lemma 1: The matrix elements $\langle B|a^j|A\rangle \equiv (\Gamma^j)^B_A$ and $\langle B|a_j|C\rangle \equiv (\Gamma_j)^B_C$ form a representation of the Clifford algebra.

Proof: This follows from $\sum |C\rangle\langle C| = I$. Namely, $\sum |C\rangle\langle C| = |0\rangle\langle 0| + \sum_i a^i|0\rangle\langle 0|a_i + \dots + a^1 \dots a^5|0\rangle\langle 0|a_5 \dots a_1$, where the sum over C runs over the 32 states shown. For any state $|A\rangle$ one has $|A\rangle = \sum |C\rangle\langle C|A\rangle$, because $\langle C|A\rangle = \delta^C_A$ by construction.

Lemma 2: The chirality matrix $\Gamma_\# = \Gamma^1\Gamma^2 \dots \Gamma^9\Gamma^0$ satisfies $\Gamma_\#^2 = 1$, and $\Gamma_\#^\dagger = \Gamma_\#$. It is given by

$$\Gamma_\# = (2a_1a^1 - 1) \dots (2a_5a^5 - 1). \quad (10.3)$$

Proof: $\Gamma_{\#} = (a^1 + a_1)(a^1 - a_1) \dots (a^5 + a_5)(a^5 - a_5)$ and $(a^1 + a_1)(a^1 - a_1) = (2a_1a^1 - 1)$. As a check note that $(2a_1a^1 - 1)^2 = 1$, and that $\{\Gamma_{\#}, a^1\} = 0$ because $\{(2a_1a^1 - 1), a^1\} = 0$. Similarly $\{\Gamma_{\#}, a_1\} = 0$. Further, $\Gamma_{\#}|0\rangle = |0\rangle$.

Lemma 3: $\langle B|a^j|C\rangle = \langle C|a_j|B\rangle = \text{real}$.

Proof: This follows from the fact that one obtains the second matrix element from the first by left-right reflection, and from the fact that the anticommutation relations have the same symmetry and are real: $\{a^k, a_l\} = \{a_l, a^k\} = \delta_l^k$.

Lemma 4: The matrix representation of $\Gamma^1, \Gamma^3, \Gamma^5, \Gamma^7, \Gamma^9$ is real and symmetric while that of $\Gamma^2, \Gamma^4, \Gamma^6, \Gamma^8$ and Γ^0 is purely imaginary and antisymmetric.

Proof: $\langle A|a^j \pm a_j|B\rangle = \langle B|\pm a^j + a_j|A\rangle$.

Lemma 5: The charge conjugation matrix C , defined by $C\Gamma^m = -\Gamma^{m,T}C$ is given by $C = -\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0 = (a_1 - a^1)(a_2 - a^2) \dots (a_5 - a^5)$. The minus sign is added for later convenience.

Proof: $\Gamma^1, \Gamma^3, \Gamma^5, \Gamma^7, \Gamma^9$ anticommute with C , while $\Gamma^2, \Gamma^4, \Gamma^6, \Gamma^8, \Gamma^0$ commute with C , the former are symmetric while the latter are antisymmetric.

Theorem I: A chiral spinor λ can be expanded as follows

$$|\lambda\rangle = \lambda_+|0\rangle + \frac{1}{2!}\lambda_{ij}a^ja^i|0\rangle + \frac{1}{4!}\lambda^i\epsilon_{ijklm}a^ja^ka^la^m|0\rangle. \quad (10.4)$$

Proof: $\Gamma_{\#}|0\rangle = |0\rangle$; hence $\Gamma_{\#}|\lambda\rangle = |\lambda\rangle$. The 16 non-vanishing components of $|\lambda\rangle$ are the projections of the ket $|\lambda\rangle$ onto the corresponding 16 bras: in particular

$$\lambda_+ = \langle 0|\lambda\rangle = \langle \lambda|0\rangle, \quad \lambda_{ij} = \frac{1}{2!}\langle 0|a_ia_j|\lambda\rangle = \frac{1}{2!}\langle \lambda|a^ja^i|0\rangle, \quad (10.5)$$

$$\lambda^i = \frac{1}{4!}\epsilon^{ijklm}\langle 0|a_ja_ka_la_m|\lambda\rangle = \frac{1}{4!}\epsilon^{ijklm}\langle \lambda|a^ja^ka^la^m|0\rangle.$$

We are now ready to solve the ten constraints $\bar{\lambda}^\alpha \Gamma_{\alpha\beta}^m \lambda^b = 0$. These relations are equivalent to the five constraints $\lambda^T C a^j \lambda = 0$ and the five other constraints $\lambda^T C a_j \lambda = 0$. They can be rewritten as follows

$$\langle \lambda|C a^j|\lambda\rangle = 0, \quad \langle \lambda|C a_j|\lambda\rangle. \quad (10.6)$$

Theorem II: $\langle A|C|B\rangle \neq 0$ iff $A^\dagger B$ is proportional to precisely $a^1a^2a^3a^4a^5|0\rangle$.

Proof: $a_j C = -C a^j$ and $a^j C = -C a_j$, Further $C|0\rangle = -a^1a^2a^3a^4a^5|0\rangle$ and $\langle 0|C = \langle 0|a^5a^4a^3a^2a^1$. Pulling all a_j in $\langle A|$ to the right of C , we obtain, up to an overall sign,

$\langle 0|CA^\dagger B|0\rangle$ and this is only non-vanishing if all a^k in $A^\dagger B$ match the a_k in $\langle 0|C$. It follows that $\langle 0|Ca^1a^2a^3a^4a^5|0\rangle = 1$.

First set of constraints

$$\begin{aligned}\langle \lambda|Ca^{i_0}|\lambda\rangle &= \langle 0|C\left(\lambda_+ + \frac{1}{2}\lambda_{ij}a^ia^j + \frac{1}{4!}\lambda^ia^ja^ka^la^m\epsilon_{ijklm}\right)a^{i_0}\lambda\rangle \\ &= 2\left(\lambda_+\lambda^{i_0} + \frac{1}{4!}\epsilon^{i_0jklm}\lambda_{jk}\lambda_{lm}\right).\end{aligned}\quad (10.7)$$

Second set of constraints

$$\begin{aligned}\langle \lambda|Ca_{i_0}|\lambda\rangle &= \langle 0|C\left(\lambda_+ + \frac{1}{2}\lambda_{ij}a^ia^j + \frac{1}{4!}\lambda^ia^ja^ka^la^m\epsilon_{ijklm}\right)a_{i_0}\lambda\rangle \\ &= -2\lambda_{i_0j}\lambda^j.\end{aligned}\quad (10.8)$$

Main Result: The solution of the first set of constraints $\lambda_+\lambda^i + \frac{1}{4!}\epsilon^{ijklm}\lambda_{jk}\lambda_{lm} = 0$ is given by

$$\lambda^i = -\frac{1}{4!\lambda_+}\epsilon^{ijklm}\lambda_{jk}\lambda_{lm}.\quad (10.9)$$

The solution automatically satisfies the second set of constraints because

$$\lambda^i\lambda_{in} = \epsilon^{ijklm}\lambda_{jk}\lambda_{lm}\lambda_{in} = 0.\quad (10.10)$$

Proof: A totally antisymmetric tensor with 6 indices in 5 dimensions vanishes. Hence $\lambda^i\lambda_{in}$ is equal to a sum of 5 terms, due to exchange n with j, k, l, m and i , respectively. Interchanging n with i yields minus the original tensor, but also interchanging n with j, k, l and m yields each time minus the original expression. Hence the expression vanishes.

Comment 1: The fact that a pure chiral spinor contains 11 independent complex components leads to a vanishing central charge in Berkovits' approach with variables x^m, θ^α and the conjugate momentum p_α , and λ^α with conjugate momentum $p_{(\lambda)\alpha}$: $c = +10_x - 2 \times 16_{\theta p} + 2 \times 11_{\lambda, p_\lambda} = 0$. In our approach we have 16 independent real component in λ^α and 16 conjugate momenta $p_{(\lambda)\alpha}$ with $\alpha = 1, \dots, 16$. Also in our case $c = 0$, but there are more ghosts, and there is nowhere a decomposition w.r.t. a subgroup of $SO(9, 1)$.

Comment 2: In the decomposition in Theorem I, one can choose all λ 's to be real, and λ^i to be expressed in terms of λ_+ and λ_{ij} as in (10.9). Then λ is a real chiral spinor. However, the Dirac matrices are complex, so under a Lorentz transformation λ becomes complex in a general Lorentz frame.

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